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THE BARNETT APPROXIMATION IN THE THEORY OF HYDRODYNAMIC FLUCTUATIONS*

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The Langevin dynamics and fluctuational-dissipative relationships for the hydrodynamic fluctuations for systems which are described in the third Barnett order with respect to the gradients of the hydrodynamic variables are generalized on the basis of a kinetic approach.

We know /1-3/ that, in the case of slow non-isothermal gas flows, the Navier-Stokes-Fourier (NSF) terms and certain Barnett terms in the hydrodynamic equations are of the same order of magnitude and form the basic asymptotic approximation to the solution to an equal extent. In the case of such flows the dynamics of the mean values of the hydrodynamic variables are still described by equations which are of the third Barnett order with respect to the gradients /1-3/. This result generates a natural "response" in the theory of non-equilibrium hydrodynamic fluctuations. In fact, the question arises in this theory as to whether the existing equations, that is, equations which have been linearized with respect to fluctuations around a stable non-equilibrium state of the NSF equation with Langevin local equilibrium sources /4/ in the Landau-Lifshitz form for the description of the fluctuations in the flows considered in /1-3/, are suitable. In order to answer this question, it is necessary to solve two problems: to construct the Langevin equations for the dynamics of the fluctuations in the third Barnett approximation with respect to the gradients and to estimate the contribution of the new terms both to the dynamical operator and to the Langevin source of fluctuations. The first of these problems will be solved below.

The results in /5, 6/ are generalized and supplemented in this paper. In particular, it will be shown that the contributions to the Landau-Lifshitz formulae, which are linear with respect to the gradients, are generated by the Barnett non-equilibrium of the system.

1. Langevin description of small fluctuations of hydrodynamic variables. Let us consider a one-component, non-equilibrium gas and write the transport equations in its hydrodynamic variables in the tensor form

$$\partial\Phi_\mu/\partial t = \Theta_\mu[\Phi], \quad \mu = 0, 1, \ldots, 4$$

Here, $\Phi_\mu = (n, u_k, e)$ is a five-dimensional vector, the components of which are the mean values of the hydrodynamic variables: n is the density, $u_k\ (k = 1, 2, 3)$ are the three components of the hydrodynamic velocity, $e = 3kT/2$ is the thermal energy (k is Boltzmann's constant and T is the temperature) and Θ_μ is the dynamic operator with the components

$$\Theta_0 = -\nabla_i n u_i, \quad \Theta_k = -u_i\nabla_i u_k - \rho^{-1}\nabla_i \Pi_{ik}$$
$$\Theta_4 = -u_i\nabla_i e - n^{-1}\nabla_i q_i - n^{-1}\Pi_{ij}\nabla_j u_i, \quad \rho = mn$$

where Π_{ij} is the pressure tensor and q_i is the thermal flux vector.

In the NSF approximation we have

$$\Pi_{ij} = \Pi_{ij}^{(1)} = p^{(0)}\delta_{ij} + p_{ij}^{(1)}, \quad q_i = q_i^{(1)}, \quad p_{ij}^{(1)} = -2\eta E_{ij}^{kl}\nabla_k u_l,$$
$$q_i^{(1)} = -\lambda\nabla_i T, \quad E_{ij}^{kl} = {}^1/_2(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) - {}^1/_3\delta_{kl}\delta_{ij}, \quad i, j, k, l = 1, 2, 3$$

where $p^{(0)}$ is the pressure, η is the viscosity and λ is the thermal conductivity. Using this notation the system of Langevin equations for the hydrodynamic fluctuations $\delta\Phi_\mu = (\delta n, \delta u_k, \delta e)$ has the form

$$\partial\delta\Phi_\mu/\partial t - \Theta_{\mu,\nu}'[\Phi]\delta\Phi_\nu = \delta G_\mu \tag{1.1}$$

Here, $\Theta_{\mu,\nu}'$ is a linearized dynamic operator, the action of which on an abitrary function of the coordinates $\varphi(r)$ has the form

$$\Theta_{\mu,\nu}'\varphi(r) = \int dr'\varphi(r')\{\delta\Theta_\mu[\Phi; r]/\delta\Phi_\nu(r')\}$$

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Here and subsequently, summation from 0 to 4 is carried out over repeated Greek indices. The inhomogeneous term δG_μ is a Langevin random Gaussian source of hydrodynamic fluctuations with a zero mean value and the components

$$\delta G_0 = 0, \quad \delta G_k = -\rho^{-1}\nabla_i \delta P_{ik}, \quad \delta G_4 = -n^{-1}\nabla_i \delta Q_i - n^{-1}\delta P_{ij}\nabla_j u_i$$

and the correlation functions

$$\begin{gathered}\langle \delta G_k(1)\,\delta G_l(2)\rangle = (\rho_1\rho_2)^{-1}\nabla_{1i}\nabla_{2j}\langle \delta P_{ik}(1)\,\delta P_{jl}(2)\rangle,\ \langle \delta G_k(1)\,\delta G_4(2)\rangle = \\ (\rho_1 n_2)^{-1}\nabla_{1i}\nabla_{2j}\langle \delta P_{ik}(1)\,\delta Q_j(2)\rangle + (\rho_1 n_2)^{-1}(\nabla_{2j}u_i)\nabla_{1l}\langle \delta P_{lk}(1)\,\delta P_{ij}(2)\rangle \\ \langle \delta G_4(1)\,\delta G_4(2)\rangle = (n_1 n_2)^{-1}\{\nabla_{1i}\nabla_{2j}\langle \delta Q_i(1)\,\delta Q_j(2)\rangle + \\ \langle \delta P_{ij}(1)\,\delta P_{kl}(2)\rangle(\nabla_{1i}u_j)(\nabla_{2l}u_k) + (\nabla_{2j}u_i)\nabla_{1l}\langle \delta Q_l(1)\,\delta P_{ij}(2)\rangle + \\ (\nabla_{1j}u_i)\nabla_{2l}\langle \delta P_{ij}(1)\,\delta Q_l(2)\rangle\}, \quad (1)\equiv(t_1, r_1), \quad (2)\equiv(t_2, r_2)\end{gathered} \tag{1.2}$$

Summation from 1 to 3 is carried out over repeated Latin indices.

The quantities δP_{ij} and δQ_i are the fluctuating components of the stress tensor $\delta p_{ij} = \delta'(p_{ij}) + \delta P_{ij}$ and the thermal flux vector $\delta q_i = \delta'(q_i) + \delta Q_i$ respectively, where δ' is a linearizing operator with respect to the fluctuations. For example,

$$\delta'(n^{-1}\nabla_i q_i) = -n^{-2}\delta n \nabla_i q_i + n^{-1}\nabla_i \delta q_i$$

It follows from expression (1.2) that the lowest order of the pair correlators of the Langevin sources is second order with respect to the gradients. A Langevin source makes a contribution of the same order to the solution of the system of Eqs.(1.1).

Actually, since the pair correlator $\langle \delta\Phi_\mu(1)\,\delta\Phi_\nu(2)\rangle$ is the required object when investigating the dynamics of Gaussian thermal fluctuations, the required proof now follows from the formal solution of system (1.1) of the form

$$\begin{gathered}\langle \delta\Phi_\mu(t, r_1)\,\delta\Phi_\nu(t, r_2)\rangle = \exp\{-t[\Theta'_{\mu,\alpha}(r_1) + \Theta'_{\nu,\beta}(r_2)]\}\langle \delta\Phi_\alpha(0, r_1)\,\delta\Phi'_\beta(0, r_2)\rangle + \\ \int_0^t d\tau \exp\{-(t-\tau)[\Theta'_{\mu,\alpha}(r_1) + \Theta'_{\nu,\beta}(r_2)]\}\langle \delta G'_\alpha(\tau, r_1)\,\delta G'_\beta(\tau, r_2)\rangle\end{gathered}$$

where the δ-like nature of the time correlation

$$\langle \delta G_\alpha(\tau_1, r_1)\,\delta G_\beta(\tau_2, r_2)\rangle = \delta(\tau_1 - \tau_2)\langle \delta G'_\alpha(\tau_1, r_1)\,\delta G'_\beta(\tau_1, r_1)\rangle$$

has been made use of.

At the present time the Langevin dynamics of the fluctuations of hydrodynamic variables have been fairly fully investigated in the equilibrium state and in the region of non-equilibrium states which are described in the NSF approximation /7, 8/. In this approximation, we have

$$\Theta'_{\mu,\nu} = \Theta^{(1)'}_{\mu,\nu}[\nabla] + \Theta^{(2)'}_{\mu,\nu}[\nabla^2]$$

where $\Theta^{(1)}_\mu$ is an Euler operator which is linear with respect to the gradients with the components

$$\Theta^{(1)}_0 = -\nabla_i n u_i, \quad \Theta^{(1)}_k = -u_i\nabla_i u_k - \rho^{-1}\nabla_k p^{(0)}$$
$$\Theta^{(1)}_4 = -u_i\nabla_i e - n^{-1}p^{(0)}\nabla_i u_i$$

and $\Theta^{(2)}_\mu$ is an NSF operator which is bilinear with respect to the gradients with the components

$$\Theta^{(2)}_0 = 0, \quad \Theta^{(2)}_k = -\rho^{-1}\nabla_l p^{(1)}_{kl}, \quad \Theta^{(2)}_4 = -n^{-1}\nabla_i q^{(1)}_i - n^{-1}p^{(1)}_{kl}\nabla_l u_k$$

and $\delta P_{ij} = \delta P^{(0)}_{ij}$, $\delta Q_i = \delta Q^{(0)}_i$ are the fluctuating components of the thermodynamic fluxes which satisfy the locally equilibrium Landau-Lifshitz formulae /7, 8/

$$\begin{gathered}\langle \delta P^{(0)}_{ij}(1)\,\delta P^{(0)}_{kl}(2)\rangle = \delta(1-2)\,E^{ij}_{hs}E^{kl}_{sg}4n^{-1}(1)\,p^{(0)}(1)\,\eta(T)\,\delta_{hg} \\ \langle \delta Q^{(0)}_i(1)\delta Q^{(0)}_j(2)\rangle = \delta(1-2)\delta_{ij}2n^{-1}(1)p^{(0)}(1)T(1)\lambda(T) \\ \langle \delta Q^{(0)}_i(1)\,\delta P^{(0)}_{kl}(2)\rangle = \langle \delta Q^{(0)}_i(1)\rangle = \langle \delta P^{(0)}_{kl}(1)\rangle = 0 \\ \delta(1-2) \equiv \delta(t_1 - t_2)\delta(r_1 - r_2)\end{gathered} \tag{1.3}$$

In a state of thermodynamic equilibrium $\Theta'_{\mu,\nu} = \Theta^{(1)'}_{\mu,\nu}|_{\Phi_\mu = \mathrm{const}}$, and it is necessary to put $\Phi_\mu = \mathrm{const}$ in formulae (1.3).

One of the possible generalizations of the linear theory of non-equilibrium hydrodynamic fluctuations involves extending the domain of its applicability beyond the limits of the NSF approximation. By analogy with the hydrodynamics of mean values, let us construct the dynamical equations for the fluctuations of the hydrodynamic variables in the domain of states which is described by the following Barnett approximation, that is, in the third order with respect to the gradients. In order to do this we shall use a kinetic approach in conjunction with a modified scheme for the Chapman-Enskog (CE) method.

2. ***Basic kinetic equations and the formulation of the problem.*** In the domain of non-equilibrium stable states of the gas, the evolution of the thermal fluctuations of the macro-density $\delta N = N - F$, $F = \langle N \rangle$ ($N(t, x)$ is the random macrodensity field, $x = (r, v)$) obeys the system of kinetic equations /7/

$$(\partial/\partial t + v_i \nabla_i)\,\delta N = J_v'(F)\,\delta N + \delta I \qquad (2.1)$$
$$(\partial/\partial t + v_i \nabla_i)\,F = J_v(F, F)$$

Here, $J_v(F, F)$ and $J_v'(F)$ are the integral and linearized Boltzmann collision operators respectively and δI is an external random source of fluctuations. The latter is a random Gaussian process with zero mean and a correlation function of the form

$$\langle \delta I(t_1, x_1)\,\delta I(t_2, x_2)\rangle = \delta(1-2)\{\delta(v_1 - v_2)\,J_{v_1}(F, F) + \qquad (2.2)$$
$$J_{v_1 v_2}(F, F) - [J_{v_1}'(F) + J_{v_2}'(F)]\,F\delta(v_1 - v_2)\} \equiv \delta(1-2)I[F, F]$$

where $J_{v_1 v_2}(F, F)$ is the unintegrated collision integral /7/.

Let us now introduce a system of additive collision invariants Ψ_μ ($\mu = 0, 1, \ldots, 4$) with the components

$$\Psi_\mu = \{1;\ n^{-1}c_k,\ k = 1, 2, 3;\ n^{-1}(^1/_2 mc^2 - e)\},\quad c_k = v_k - u_k$$

With the aid of this system, we represent the fluctuations of the hydrodynamic variables in the form

$$\delta n(t, r) = \int dv \Psi_0 \delta N, \qquad \delta u_k(t, r) = \int dv \Psi_k \delta N \qquad (2.3)$$
$$\delta e(t, r) = \int dv \Psi_4 \delta N, \qquad \delta N = \delta N(t, x)$$

We obtain the system of transport equations for the hydrodynamic fluctuations by calculating the moments of the first equation of (2.1) in velocity space. By taking account of expression (2.3) and the orthogonality of the realizations of the random field δI to the collision invariants ($\int dv \Psi_\mu \delta I = 0$), we find

$$\partial \delta n/\partial t + \delta'(\nabla_i n u_i) = 0 \qquad (2.4)$$
$$\partial \delta u_k/\partial t + \delta'(u_i \nabla_i u_k) = -\delta'(\rho^{-1}\nabla_i \Pi_{ik})$$
$$\partial \delta e/\partial t + \delta'(u_i \nabla_i e) = -\delta'(n^{-1}\nabla_i q_i + n^{-1}\Pi_{ij}\nabla_j u_i) +$$
$$n^{-1}\nabla_i(\delta u_j)(\Pi_{ij} + {}^3/_2 p^{(0)}\delta_{ij})$$

The mean values of the thermal flux, the pressure tensor and their fluctuations are defined by the expressions

$$q_k(t, r) = \int dv\,^1/_2 mc^2 c_k F(t, x), \qquad \delta q_k(t, r) = \int dv\,^1/_2 mc^2 c_k \delta N(t, x) \qquad (2.5)$$
$$\Pi_{kl}(t, r) = \int dv\, m c_k c_l F(t, x), \qquad \delta \Pi_{kl}(t, r) = \int dv\, m c_k c_l \delta N(t, x)$$

For the closure of the transport Eqs.(2.4), it is necessary to solve two problems: to find the functional dependence of q_k and Π_{kl} on Φ_μ and the functional dependence of δq_k and $\delta \Pi_{kl}$ on Φ_μ and $\delta \Phi_\mu$. The first of these problems is solved by constructing the normal solutions of the Boltzmann equation (the second equation of (2.1)). The Chapman-Enskog method, up to the third order terms with respect to the gradients, yielded the Fourier and Newton relationships and the Barnett additions to them. We use the same method to solve the second problem. In order to do this, we pass, in Eqs.(2.1), to the dimensionless variables and functions which are characteristic for the hydrodynamic regime

$$t_G = twL^{-1},\ r_G = rL^{-1},\ v_G = vw^{-1},\ F_G = Fw^3 n^{-1}$$

(L is the characteristic hydrodynamic scale of length and w is the mean thermal velocity of a particle). The dimensions of the Gaussian fields are determined in terms of their second moments /5/

$$\delta N_G = \delta N w^{-3}(L^3 n^{-1})^{-1/2},\quad \delta I_G = \delta I\,[L^2 w^2 (l n^{-1})^{1/2}]^{-1}$$

(l is the mean free path).

A small parameter $\varepsilon = lL^{-1}$, the Knudsen number, appears in the dimensionless variables in Eqs.(2.1)

$$\varepsilon (\partial/\partial t + v_i\nabla_i)\delta N = J_v' (F)\delta N + \sqrt{\varepsilon}\delta I \qquad (2.6)$$
$$\varepsilon (\partial/\partial t + v_i\nabla_i)F = J_v (F, F)$$

Here and subsequently, we omit the index G accompanying the variables and functions. When $\varepsilon \ll 1$, the Chapman-Enskog method enables us to construct a class of normal asymptotic solutions of the system of Eqs.(2.6) of the form

$$\delta N (t, x) = \delta N [\Phi (t), \delta\Phi (t); x], \quad F (t, x) = F [\Phi (t); x]$$

and, on the basis of this system, to calculate the fluxes using formulae (2.5) and, thereby, to close the transport Eqs.(2.4).

Before proceeding with the search for the normal solutions, let us first made an essential modification to the scheme of the Chapman-Enskog method.

The essence of the Chapman-Enskog method lies in the fact that, under a hydrodynamic regime, not only the required function but also the evolutionary operator of the kinetic equation are expanded in series in the Knudsen number ε /9/

$$\partial/\partial t = d_{(0)}/\partial t + \varepsilon\partial_{(1)}/dt + \varepsilon^2 \partial_{(2)}/\partial t \qquad (2.7)$$

The Chapman-Enskog method can also be used in the kinetic theory of non-equilibrium hydrodynamic fluctuations. When this is done, however, an expansion of the evolutionary operator which is different from (2.7) turns out to be convenient. Actually, unlike the usual homogeneous equations of hydrodynamics, the Langevin equations for the fluctuations are inhomogeneous equations. Moreover, in the general case, the inhomogeneous terms in these equations have increasing orders of the mean values of the hydrodynamic variables with respect to the gradients. On account of this the additional problem of successively taking the inhomogeneous terms into account arises within the framework of the standard scheme of the Chapman-Enskog method based on the expansion (2.7).

The automatic ordering and allowance for the inhomogeneous terms of the hydrodynamic equations occurs with the modified expansion of the evolutionary operator

$$\varepsilon\partial / \partial t = \varepsilon\partial_{(1)}/\partial t + \varepsilon^2\partial_{(2)}/\partial t + \varepsilon^3\partial_{(3)}/\partial t \qquad (2.8)$$

This expansion generates a gradient expansion of the hydrodynamic equations, unlike the expansion (2.7) which generates a gradient expansion of the thermodynamic fluxes, and, furthermore, the parameter ε determines the degree of inhomogeneity of the system.

By applying the Chapman-Enskog method with the expansion (2.8) to the Boltzmann equation, we obtain Euler's equations in the first order with respect to ε, NSF equations in the second order with respect to ε, the Barnett equations in the third order of ε and so on.

3. Hydrodynamic asymptotic behaviour of the solutions of the system of equations (2.6) in the Barnett approximation. The asymptotic expansion of the well-known solution of the Boltzmann equation in series in ε leads to the corresponding expansion of the collision operator and the source term

$$F = \sum_{n=0}^{\infty} \varepsilon^n F^{(n)} [\Phi; x], \quad J_v' (F) = \sum_{n=0}^{\infty} \varepsilon^n J_n' (F^{(n)}) \qquad (3.1)$$
$$\delta I = \sum_{n=0}^{\infty} \varepsilon^{n/2}\delta I^{(n/2)}$$

where the Gaussian field pair correlator $\delta I^{(n/2)}$ is determined from expression (2.2). By virtue of the statistical independence of the individual terms of series (3.1), we have

$$\langle\delta I (1, v_1) \delta I (2, v_2)\rangle = \sum_{n,\, m=0}^{\infty} \varepsilon^{(n+m)/2} \langle\delta I^{(n/2)}\delta I^{(m/2)}\rangle =$$
$$\delta (1 - 2) \sum_{n',\, m'=0}^{\infty} \varepsilon^{(n'+m')} I [F^{(n')}, F^{(m')}]$$

Here,

$$\langle\delta I^{(n/2)}\delta I^{(m/2)}\rangle = \delta_{nm}\delta (1 - 2) \sum_{k=0}^{n} I [F^{(k)}, F^{(n-k)}] \qquad (3.2)$$

The asymptotic behaviour of the solutions of the first equation of (2.6) when $\varepsilon \ll 1$ was matched with the formal expansion

$$\delta N(t,x)=\sum_{n=0}^{\infty}\varepsilon^{n/2}\delta N^{(n/2)}[\Phi(t),\delta\Phi(t);x] \tag{3.3}$$

In order to determine the coefficients of series (3.3), we employ a modified Chapman-Enskog scheme, in accordance with which

$$\varepsilon\partial/\partial t=\sum_{n=2}^{\infty}\varepsilon^{n/2}\partial_{(n/2)}/\partial t$$

The technique used to find the terms of this series is based on the use of the conditions for the non-decomposability of the hydrodynamic variables

$$\int dv\Psi_0\delta N^{(k/2)}=\delta_{k0}\delta n,\quad \int dv\Psi_4\delta N^{(k/2)}=\delta_{k0}\delta e$$
$$\int dv\Psi_l\delta N^{(k/2)}=\delta_{k0}\delta u_l;\quad l=1,2,3;\quad k=0,1,2,\ldots$$

and the conditions for the solvability of the integral equations for $\delta N^{(k/2)}$.

After the determination of the first M terms of series (3.3), the results of the method yield the expressions

$$\delta q_i=\sum_{n=0}^{M}\delta q_i^{(n/2)}\equiv\sum_{n=0}^{M}\int dv^1/_2mc^2c_i\delta N^{(n/2)} \tag{3.4}$$
$$\delta\Pi_{kl}=\sum_{n=0}^{M}\delta\Pi_{kl}^{(n/2)}\equiv\sum_{n=0}^{M}\int dvmc_kc_l\delta N^{(n/2)}$$

which enable one to close the transport Eqs.(2.4).

Next, according to the Chapman-Enskog method, it is necessary to substitute the expansion (3.1)-(3.3) and the series for $\varepsilon\partial/\partial t$ into the first equation of (2.6) and, when this is done, to obtain the equations for $\sigma N^{(n/2)}$ $(n=0,1,2,\ldots)$ and to solve them using the well-known expressions for the functions $F^{(n)}$. The implementation of this program is simple, but it is rather large and, from a procedural point of view, does not differ in any way from the known implementation in /5/ and /10/. We shall therefore only present the final result.

The expressions for the first five terms of the series (3.3) have the form

$$\delta N^{(0)}=(\delta\Phi_\mu;\partial_{\Phi_\mu})F^{(0)}\equiv DF^{(0)},\quad D=(\delta\Phi_\mu;\partial_{\Phi_\mu}) \tag{3.5}$$
$$\delta N^{(1/2)}=-[J_v'(F^{(0)})]^{-1}\delta J^{(0)},\quad \delta N^{(1)}=DF^{(1)}-[J_v'(F^{(0)})]^{-1}\delta J^{(1/2)},$$
$$\delta N^{(3/2)}=-[J_v'(F^{(0)})]^{-1}\{\delta J^{(1)}-(\partial_{(3/2)}\delta\Phi_\mu/\partial t;\partial_{\Phi_\mu})F^{(0)}\}$$
$$\delta N^{(2)}=DF^{(2)}+[J_v'(F^{(0)})]^{-1}\{-\delta J^{(3/2)}+J_v'(F^{(1)})DF^{(1)}+$$
$$(\partial_{(2)}\delta\Phi_\mu/\partial t-\Theta_{\mu,\nu}^{(2)'}\delta\Phi_\nu;\partial_{\Phi_\mu})F^{(0)}\}$$

Here, use has been made of the notation

$$\delta J^{(0)}=\delta I^{(0)},\quad \delta J^{(1/2)}=\delta I^{(1/2)},\quad \delta J^{(1)}=\delta I^{(1)}- \tag{3.6}$$
$$\{J_v'(F^{(1)})-(\partial_{(1)}/\partial t+v_i\nabla_i)\}[J_v'(F^{(0)})]^{-1}\delta I^{(0)}$$
$$\delta J^{(3/2)}=\delta I^{(3/2)}-\{J_v'(F^{(1)})-(\partial_{(1)}/\partial t+v_i\nabla_i)\}[J_v'(F^{(0)})]^{-1}\delta I^{(1/2)}$$

and ∂_{Φ_μ} for the functional derivative $\delta/\delta\Phi_\mu$. The symbol $(\cdot;\cdot)$ denotes the inner product of the functions enclosed in the brackets, for example,

$$(\delta\Phi_\mu;\partial_{\Phi_\mu})\varphi(r)=\int dr'\delta\Phi_\mu(r')\{\delta\varphi(r)/\delta\Phi_\mu(r')\}$$

By substituting expressions (3.5) into formulae (3.4), after evaluating the integrals in velocity space when $M=4$, we get

$$\delta q_k=-D(q_k^{(1)}+q_k^{(2)})+(^5/_2p^{(0)}\delta_{kl}+p_{kl}^{(1)}+p_{kl}^{(2)})\delta u_l+\delta Q_k+\delta q_k', \tag{3.7}$$
$$\delta\Pi_{kl}=-D(p^{(0)}\delta_{kl}+p_{kl}^{(1)}+p_{kl}^{(2)})+\delta P_{kl}+\delta p_{kl}',$$
$$\delta Q_k=\sum_{n=0}^{3}\delta Q_k^{(n/2)},\quad \delta P_{kl}=\sum_{n=0}^{3}\delta P_{kl}^{(n/2)}$$

$$\delta Q_k^{(n/2)} = -\int dv^1/_2 mc^2 c_k [J_v'(F^{(0)})]^{-1} \delta J^{(n/2)} = n^{-1}kT \int dv A_k \delta J^{(n/2)},\quad (3.8)$$
$$\delta P_{kl}^{(n/2)} = -\int dv mc_k c_l [J_v'(F^{(0)})]^{-1} \delta J^{(n/2)} = n^{-1}kT \int dv B_{kl} \delta J^{(n/2)}$$
$$A_k = -[J_v'(F^{(0)})]^{-1} nc_k (mc^2/(2kT) - {}^5/_2)$$
$$B_{kl} = -[J_v'(F^{(0)})]^{-1} (\rho/(kT)) E_{ij}^{kl} c_i c_j$$

Here, $q_k^{(2)}$ and $p_{kl}^{(2)}$ are the Barnett additions to the thermal flux and the stress tensor /9/. The quantities $\delta q_k'$ and $\delta p_{kl}'$ are generated by the third term in $\delta N^{(2)}$ and are expressed in terms of the so-called modified integral brackets of the phase functions A_k and B_{kl}. Such brackets can be approximately calculated by expanding the functions A_k and B_{kl} in series in Sonin polynomials /9/. When this is done, it turns out that, to a first approximation with respect to the polynomials, the modified brackets, which form $\delta q_k'$ and $\delta p_{kl}'$, are equal to zero.

Hence, in the third order with respect to the gradients of the hydrodynamic variables, the transport Eqs.(2.4), when account is taken of formulae (3.7) and the expressions

$$\Pi_{kl} = p^{(0)}\delta_{kl} + p_{kl}^{(1)} + p_{kl}^{(2)}, \qquad q_k = q_k^{(1)} + q_k^{(2)}$$

are transformed into the system of linearized hydrodynamic equations in the Barnett approximation

$$\partial\delta\Phi_\mu/\partial t = \Theta_{\mu,\nu}^{(B)'}\delta\Phi_\nu + \delta G_\mu, \quad \Theta_\mu^{(B)} = \Theta_\mu^{(1)} + \Theta_\mu^{(2)} + \Theta_\mu^{(3)} \quad (3.9)$$

Here, $\Theta_\mu^{(B)}$ is the dynamic operator of the Barnett approximation, the quantities $\Theta_\mu^{(1)}$ and $\Theta_\mu^{(2)}$ have been defined previously and $\Theta_\mu^{(3)}$ and δG_μ have the following components

$$\Theta_0^{(3)} = 0, \quad \Theta_k^{(3)} = -\rho^{-1}\nabla_i p_{ik}^{(2)}, \quad \Theta_4^{(3)} = -n^{-1}\nabla_i q_i^{(2)} - n^{-1} p_{ij}^{(2)} \nabla_j u_i,$$
$$\delta G_0 = 0, \quad \delta G_k = -\rho^{-1}\nabla_i \delta P_{ik}, \quad \delta G_4 = -n^{-1}\nabla_i \delta Q_i - n^{-1}\delta P_{ij}\nabla_j u_i$$

The Gaussian properties and the space-time δ-correlation property of the quantities δP_{ij} and δQ_i follow from formulae (3.8) and the statistical properties of the random fields $\delta J^{(n/2)}$.

The problem of finding the pair correlators of the components of the Langevin source δG_μ in the Barnett approximation (the third approximation of the hydrodynamic variables with respect to the gradients) when account is taken of formulae (1.2) reduces to the calculation of the pair correlators of the quantities δP_{ij} and δQ_i, apart from terms of the first order with respect to the gradients, since, according to (1.2), each of the components of the correlator $\langle\delta G_\mu(1)\delta G_\nu(2)\rangle$ is already of second order with respect to the gradients.

In the Barnett approximation when account is taken of expression (3.2), representation (3.8) for the correlator of the fluctuating component of the thermal flux yields

$$\langle\delta Q_k(1)\,\delta Q_l(2)\rangle = \langle\delta Q_k(1)\,\delta Q_l(2)\rangle^{(B)} \equiv \quad (3.10)$$
$$\langle\delta Q_k^{(0)}(1)\,\delta Q_l^{(0)}(2)\rangle + \langle\delta Q_k(1)\,\delta Q_l(2)\rangle^{(1)}$$
$$\langle\delta Q_k(1)\,\delta Q_l(2)\rangle^{(1)} = \langle\delta Q_k^{(1/2)}(1)\,\delta Q_l^{(1/2)}(2)\rangle + \quad (3.11)$$
$$\langle\delta Q_k^{(0)}(1)\,\delta Q_l^{(1)}(2)\rangle + \langle\delta Q_k^{(1)}(1)\,\delta Q_l^{(0)}(2)\rangle$$
$$\langle\delta Q_n^{(n/2)}(1)\,\delta Q_l^{(m/2)}(2)\rangle = \frac{k^2 T_1 T_2}{n_1 n_2}\int dv_1\, dv_2 A_k(v_1) \times \quad (3.12)$$
$$A_l(v_2)\langle\delta J^{(n/2)}(1, v_1)\,\delta J^{(m/2)}(2, v_2)\rangle$$

Expressions which are similar in form can be obtained for the correlators $\langle\delta P_{kl}(1)\,\delta P_{ij}(2)\rangle$ and $\langle\delta P_{kl}(1)\,\delta Q_i(2)\rangle$.

The first term in (3.10) is of zero order with respect to the gradients of the hydrodynamic variables while the second term is of the first order. We note that the linear contributions in the Landau-Lifshitz formulae obtained in /5/ and /6/ simply reduce to the first term in expressions of the type of (3.11) although, as follows free formulae (3.12), (3.6) and (3.2), all three terms of this expression are of the same order of smallness. In its turn, it follows from this that, with the aid of the modified scheme for the Chapman-Enskog method, it is possible to take account of the inhomogeneous terms in the Langevin hydrodynamic equations more fully than was done when the standard implementation of the method was used /5, /6.

According to formula (3.12), the calculation of the pair correlators reduces to the evaluation of the integrals in velocity space. These integrals are reduced with the help of formulae (3.6) and (3.2), in a similar manner to that used in /5, 10/ to two types of integral brackets of kinetic theory. For instance, in the case of the correlator (3.10), we have

$$\langle\delta Q_k(1)\,\delta Q_l(2)\rangle^{(B)} = \delta(1-2)\, n^{-1}(1)\, p^{(0)}(1)\, T(1)\, \lambda(T) \times \quad (3.13)$$

$$[\delta_{kl} + K_{kl}^{ij} p_{ij}^{(1)}(1)/p^{(0)}(1)], \qquad 4\lambda\eta\,(n/p^{(0)})^2\, T K_{kl}^{ij} = [A_k; B_{ij}A_l] +$$
$$[A_l; B_{ij}A_k] + [B_{ij}; A_kA_l] - [A_k; B_{ij}, A_l]^* - [A_l; B_{ij}, A_k]^* - [B_{ij}; A_k, A_l]^*$$

Similarly, for the other two correlators we find

$$\langle \delta P_{kl}(1)\,\delta P_{ij}(2)\rangle = \langle \delta P_{kl}(1)\,\delta P_{ij}(2)\rangle^{(B)} = \tag{3.14}$$
$$\delta(1-2)\,4\,(n(1))^{-1} p^{(0)}(1)\,\eta(T)\,[E_{ij}^{kl} + K_{kl,\,ij}^{sd} p_{sd}^{(1)}(1)/p^{(0)}(1)]$$

$$8\,(\eta n/p^{(0)})^2\, K_{kl,\,ij}^{sd} = [B_{kl}; B_{sd}B_{ij}] + [B_{ij}; B_{sd}B_{kl}] + [B_{sd}; B_{kl}B_{ij}] -$$
$$[B_{kl}; B_{sd}, B_{ij}]^* - [B_{ij}; B_{sd}, B_{kl}]^* - [B_{sd}; B_{ij}, B_{kl}]^*$$
$$\langle \delta Q_k(1)\,\delta P_{ij}(2)\rangle = \langle \delta Q_k(1)\,\delta P_{ij}(2)\rangle^{(B)} = \delta(1-2)\,(n(1))^{-1}\,\eta(T)\, K_{k,\,ij}^{l} q_l^{(1)},$$
$$\lambda\eta T\,(np^{(0)})^2\, K_{k,\,ij}^{l} = [A_k; A_lB_{ij}] + [A_l; A_kB_{ij}] + [B_{ij}; A_kA_l] - [A_k; A_l, B_{ij}]^* - \tag{3.15}$$
$$[A_l; A_k, B_{ij}]^* - [B_{ij}; A_k, A_l]^*$$

The functions A_k and B_{kl} were introduced earlier on while the conventional $[\cdot, \cdot]$ and modified $[\cdot; \cdot, \cdot]^*$ integral brackets of the derivatives of the velocity functions A, B and C are defined in the following manner /5, 10/

$$[A; B] = -n^{-2}\int dv B(v)\, J_v'(F^{(0)})\, F^{(0)} A(v)$$
$$[A; B, C]^* = -n^{-2}\int dv_1\, dv_2 B(v_1)\, C(v_2)\, J'_{v_1v_2}(F^{(0)})\, F^{(0)} A =$$
$$-n^{-2}\int dv A(v)\,\{J_v(F^{(0)}B, F^{(0)}C) + J_v(F^{(0)}C, F^{(0)}B)\}$$

The first terms in the square brackets of formulae (3.13) and (3.14), which are generated by the first term in expressions of the type (3.10), correspond to the NSF approximation and yield the classical formulae (1.3). The second terms in the square brackets of formulae (3.13) and (3.14) and formula (3.15) are generated by the second terms of formulae of the typ of (3.10) and correspond to the Barnett approximation. Formulae (3.13)-(3.15) are fluctuation-dissipative relationships for the hydrodynamic stage of the evolution of a gas in the Barnett approximation.

The overall structure of the correlators (3.13)-(3.15) is universal and valid for any interparticle interaction potentials in a gas which are consistent with the condition for the existence of a collision integral. The values of the tensor coefficients K_{kl}^{ij}, $K_{kl,ij}^{sd}$ and $K_{k,ij}^{l}$ depend on the form of the interparticle interaction potential. In the general case, it is not possible to carry out a calculation of the integral brackets which form these coefficients. In kinetic theory, integral brackets are calculated approximately by using expansions of the functions which form them in series in polynomials /9/. The technical execution of such calculations is quite massive and we shall therefore only present the result of the calculation of the above-mentioned coefficients to a first approximation of the expansion of the functions A_k and B_{ij} in series in Sonin polynomials which corresponds to the exact result in the case of a gas consisting of Maxwellian molecules

$$K_{kl}^{ij} = \frac{63}{20} E_{kl}^{ij}, \qquad K_{kl,\,ij}^{sd} = 3E_{sh}^{kl}E_{hd}^{ij}, \qquad K_{k,\,ij}^{l} = \frac{63}{5} E_{ij}^{kl}$$

The foregoing discussion allows the conclusion to be drawn that Barnett non-equilibrium of a hydrodynamic system has an effect on the process of the generation of fluctuations in the form of a contribution to the Landau-Lifshitz formulae which is linear in the gradients.

Eqs.(3.9), which have been obtained, and formulae (3.13)-(3.15) can be used to investigate the dynamics of fluctuations in slow non-isothermal flows of the continuous media considered in /1-3/.

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MODEL OF A WEAKLY NON-LOCAL RELAXING COMPRESSIBLE MEDIUM*

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A model of a weakly non-local relaxing medium with viscous dispersion is considered. The relaxation kinetics are described by a Ginzburg-Landau /1/ equation which has been generalized to the case of a compressible medium. The special features of the propagation of planar acoustic waves in the medium are studied. The latter medium has an internal time scale which arises from the description of the relaxation kinetics and a spatial scale which characterizes the degree of the non-localness of the medium. General methods for constructing models of equilibrium non-local media have been developed in /2-5/. The generalization of these methods to the case of a relaxing medium enables one to describe the structure of a non-equilibrium phase discontinuity and to calculate the dissipation on the conversion front /6/.

1. Let us assume that the internal energy u of a unit mass is a function of the system of parameters

$$s, \rho g^{ij}, \xi_\alpha, \xi_\alpha^{\cdot}, \nabla_i\xi_\alpha, \nabla_j\nabla_i\xi_\alpha, \ldots \tag{1.1}$$

where s is the entropy per unit mass of the medium, ρ is the density, g^{ij} are the contravariant components of the metric tensor in the Euclidean Eulerian system of coordinates $x^i\ (i=1,2,3)$, $\xi_\alpha^{\cdot}(\alpha=1,\ldots,n)$ are additional scalar parameters (internal degrees of freedom), the total derivative with respect to time is denoted by a dot and ∇_i is a covariant derivative in the coordinate system x_i. The thermal influx equation can be written in the form /3, 6, 7/

$$du = \rho^{-1}(-pg^{ij} + \tau^{ij} + \sigma^{ij})\nabla_j v_i\, dt - \rho^{-1}\nabla_k(q^k + Q^k)\, dt \tag{1.2}$$

where p is the pressure, τ^{ij} are the components of the viscous stress tensor, v_i are the components of the velocity vector of the medium, q^k are the components of the thermal flux vector, Q^k are the components of the vector describing the flux of non-thermal forms of energy, σ^{ij} are the components of the reactive stress tensor and Q^k and σ^{ij} are functions

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